1. Explain the following statements are true.

   (a) If $G/Z(G)$ is cyclic, then G is abelian.

   Solution: Let $G/Z(G)$ be generated by $g_0Z(G)$ and $g, h \in G$. Then, $g = g_0^m z$ and $h = g_0^n w$ for some natural numbers m, n and $z, w \in Z(G)$.

   Now, $gh = g_0^m z g_0^n w = g_0^{m+n} zw = g_0^{n+m} wz = hg$ which means G is abelian.

   (b) Every characteristic subgroup is normal.

   Solution: Let H be a characteristic subgroup and $x \in G$. Consider the inner automorphism Ad_x. As H is characteristic subgroup therefore, $Ad_x(H) \subseteq H$ which implies that H is normal.

   (c) Every abelian group of order pq where p, q are distinct primes, is cyclic.

   Solution: Let $g, h \in G$ of order p and q respectively. As G is abelian order of gh is pq which means that G is cyclic.

   (d) $Inn(S_n)$, the group of inner automorphisms of S_n is isomorphic to S_n for $n \geq 3$.

   Solution: For any group G one can show that $Inn(G) \cong G/Z(G)$.

   Define a map $\Phi : G \to Inn(G)$ by $\Phi(g) := Ad_g$ for $g \in G$. Clearly, Φ is group homomorphism and injective with kernel $Z(G)$.

   Then, using first isomophism theorem we get that $G/Z(G) \cong Inn(G)$. Now, as $Z(S_n) = id$ for $n \geq 3$ so we are done.

   (e) $Z(S_n) = 1$ for $n \geq 3$.

   Solution: Suppose $\sigma \in S_n$ is not identity then there is some i such that $\sigma(i) = j \neq i$. Pick $k \notin \{i, j\}$ and let $\tau = (j, k)$, then, $\tau\sigma \neq \sigma\tau$.

2. Let G be a group and N a normal subgroup of G. Show that there exists a bijective correspondence between subgroups of G containing N and subgroups of G/N.

   Solution: Let $Sub(G, N)$ be the set of subgroups of G containing N and $Sub(G/N)$ be the set of subgroups of G/N.

   Define, a map $\Phi : Sub(G, N) \to Sub(G/N)$ by $\Phi(H) := H/N$. This map is clearly well defined. Let $H_1 \neq H_2 \in Sub(G, N)$ then without loss of generality let $x \in H_1$ but $x \notin H_2$. So, $xN \in H_1N$ but $xN \notin H_2N$ i.e. $H_1N \neq H_2N$ and Φ is one-one. To prove that Φ is onto, let A be a subgroup of G/N and let $S := \{g \in G : gN \in A\}$. Then, $\Phi(S) = a$ which shows that Φ is onto.

3. Let n be a positive integer. Describe the group $Aut(Z_n)$, where Z_n is the cyclic group of order n.

   Solution: Let $U(n)$ be the set of all integers which are less than n and prime to n. It forms a group with respect to multiplication modulo n. Generators of Z_n are the elements of $U(n)$.

   For a generator $b \in Z_n$ we define a map $\tau_b : Z_n \to Z_n$ by $\tau_b(m) = mb$ for $m \in Z_n$. Then, τ_b becomes an automorphism of Z_n.

   Define, $\tau : U(n) \to Aut(Z_n)$ by $\tau(b) := \tau_b$ which is well defined and one one. To prove that τ is onto also, let $\phi \in Aut(Z_n)$ and $g \in Z_n$. Then, $\phi^{-1}(g) = k \implies g = \phi(k) = k\phi(1)$ i.e. $\phi(1)$ is a generator and $\phi = \tau_{\phi(1)}$. Hence, τ is onto.

4. Prove that if G is a finite group, and p is the smallest prime dividing the order of G, then any subgroup of index p is normal.

   Solution: Let H be a subgroup of index p. Then the group G acts on the left cosets G/H by left multiplication.

   It induces the permutation representation $\rho : G \to S_p$.

   Let $K = \ker \rho$ be the kernel of ρ. Since $kH = H$ for $k \in K$, we have $K \subset H$. Let $[H : K] = m$.

   By the first isomorphism theorem, the quotient group G/K is isomorphic to the subgroup of S_p, thus $[G : K]$ divides $|S_p| = p!$ by Lagrange's theorem. Since $[G : K] = [G : H][H : K] = pm$, we have pm divides $p!$ and hence m divides $(p-1)!$. If m has a prime factor q, then $q \geq p$ since the minimality of p but the factors of $(p-1)!$ are only prime numbers less than p. This implies that $m = 1$, hence $H = K$. Therefore H is normal since a kernel is always □

5. (a) Deduce Class Equation.

   Solution: Let G be a group and $a \in G$. Conjugacy class $cl(a)$ of a is defined as $cl(a) : \{gag^{-1} : g \in G\}$ and centralizer $C(a)$ of a is defined as $C(a) := \{g \in G : gag^{-1} = a\}$.

   Consider the function $T : G/C(a) \to cl(a)$ defined by $T(xC(a)) = xax^{-1}$. A routine calculation shows that T is well defined, is one-to-one, and maps the set of left cosets onto the conjugacy class of a. Thus, the number of conjugates of a is the index of the centralizer of a.

   Since the conjugacy classes partition a group, we get the Class equation $|G| = \sum |G : C(a)|$, where the sum runs over one element a from each conjugacy class of G.

   (b) Show that a group of order p^n, where p is a prime and $n \geq 1$ has a nontrivial center.

   Solution: First observe that $cl(a) = \{a\}$ if and only if $a \in Z(G)$. Thus, we may write the class equation in the form $|G| = |Z(G)| + \sum |G : C(a)|$, where the sum runs over representatives of all conjugacy classes with more than one element (this set may be empty). But $|G : C(a)| = |G|/|C(a)|$, so each term in $\sum |G : C(a)|$ has the form p^k with $k \geq 1$. Hence, $|G| - \sum |G : C(a)| = |Z(G)|$, where each term on the left is divisible by p. It follows, then, that p also divides $|Z(G)|$, and hence $|Z(G)| > 1$.

6. Show that the order of the centralizer $C_{S_n}((12)(34)$ is $(n-4)! \times 8$, for all $n \geq 4$. Determine the elements explicitly.

   Solution: Let $g := (12)(3,4)$ then $\sigma g \sigma^{-1} = (\sigma(1)\sigma(2))(\sigma(3)\sigma(4))$ for $\sigma \in S_n$. Now, $\sigma g \sigma^{-1} = g$ implies that $(\sigma(1), \sigma(2))$ can be (12) or,(21) or, (34) or, (43) and also $(\sigma(3)\sigma(4))$ can be (12) or,(21) or, (34) or, (43) but $(\sigma(1), \sigma(2)) \neq (\sigma(3)\sigma(4))$. Hence, there are 8 cases. In each cases $(n-4)$ position of σ can be filled by rest $(n-4)$ numbers in $(n-4)!$ ways. So, there are $(n-4)! \times 8$ number of elements in $C_{S_n}((12)(34)$.